Direct perturbation theory for dark solitons

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Perturbation theory for dark solitons governed by the nonlinear Schrödinger equation is developed. Orthogonality and closure of the basis of squared Jost functions are proved. The general form of a correction to the one-soliton solution up to first order is obtained. Two examples related to perturbed dynamics of dark optical solitons are considered within the framework of the adiabatic approximation.

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I. INTRODUCTION

Despite the fact that, at present, perturbation theory for solitons is well developed, unremitting interest in it exists since including diversity of perturbations into consideration brings the mathematical statement of problems closer to physical reality. Two basic approaches in the perturbation theory can be singled out. The first one, originated by papers [1,2] (see also [3,4]), is based on studying temporal evolution of scattering data associated with the perturbed nonlinear evolution equation. Another approach deals directly with a linearized equation for the first order addendum. In that way, the inverse scattering technique plays an important role, as well: it serves as a powerful tool for finding a Green's function of the corresponding linearized equation. Progress of this approach was connected with the paper [5] and with a number of consequent investigations [6–8] devoted to properties of squared Jost functions. Following Ref. [9], where one can find results related to some classical nonlinear evolution systems and obtained within the framework of the approach mentioned, we will refer to this method as a *direct* perturbation theory.

The principle point of this approach can be briefly outlined as follows. The direct spectral problem associated with the nonlinear evolution equation is investigated and appropriate combinations of squared Jost functions are defined. As they are eigenfunctions of a respective linear operator orthogonality relations can be found. Then by studying the mutual relations among small variations of the scattering data and the nonlinear field, which now plays a role of a potential, the closure of the chosen basis is proved. It allows one to rewrite a solution of a linearized equation in a form of an expansion on the basis of squared Jost functions.

It is worthwhile to mention that the squared Jost functions are of interest for many reasons [10,11,6–8], besides being the background for the direct perturbation theory. To the best of the authors' knowledge, they have not been studied yet for the stable nonlinear Schrödinger equation (NSE) subject to nonzero boundary conditions. That is why it is one of the purposes of this work to state properties of the respective squared Jost functions.

Another aim of the present paper is to develop the direct perturbation theory for a dark soliton dynamics governed by a perturbed NSE

$$iq_t + q_{xx} + 2(\rho^2 - |q|^2)q = \mu f(x,t) R[q] , \tag{1}$$

where μ is a small parameter, $\mu \ll 1$, $f(x,t)$ is an arbitrary bounded function, and $R[q]$ is a functional on q (in order to shorten some designations in what follows, the right hand side of Eq. (1) will be set as $\hat{R}$). Equation (1) is subject to the boundary conditions

$$q \to \rho \text{ at } x \to -\infty ; q \to \rho \varepsilon^2 \text{ at } x \to \infty \tag{2}$$

($\varepsilon = e^{i\vartheta/2}$ and ρ and $\vartheta \in [0,\pi]$ are constants). Clearly, for these boundary conditions to be coordinated with (1) one has to require decay of the right hand side of (1) with x (it does not matter whether it is due to $f(x,t)$ or $R[q]$). If $\mu = 0$, (1) reduces to the conventional NSE, a soliton solution of which has a form [12]

$$q_s = \rho \frac{1+\varepsilon^2 \exp[\nu(x - vt - x_0)]}{1+\exp[\nu(x - vt - x_0)]} , \tag{3}$$

with $\nu = \omega \sin(\vartheta/2)$, $v = -\omega\cos(\vartheta/2)$, and $\omega = 2\rho$.

The direct perturbation theory of a bright NSE soliton can be found in [9].

Particular cases of perturbed dynamics of the dark soliton within the framework of Eq. (1) have already been considered in the literature. In the context of optical applications, the attention of the preceding studying was paid mainly to the effect of dissipation, i.e., when $f(x,t)R[q] \equiv iq$ [13], and periodical amplification [14]. The influence of some other physical factors on dark soliton dynamics has been treated analytically in the small amplitude limit [15]. Results on the perturbed dark soliton dynamics governed by the generalized NSE [16] can be found in [17]. There the reduction of the perturbed NSE to the conventional NSE with modified initial conditions, similar to that used in [16] for a bright soliton case, has been employed. However, the last method is appropriate only for a special kind of perturbation and hardly can be extended to a generic case. The generalization of [16] to the dark soliton case when $f(x,t)R[u] \equiv q(|q|^2)_x$ and the respective analysis of the perturbed dynamics have been reported recently in [18].

While most of the calculations about dark soliton dy-

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namics were made on the basis of the numerical study, development of the consistent analytical theory (similar to those existing for bright solitons of different equations) was set aside.

Here we construct a perturbative expansion of $q(x,t)$ about $q_s(x,t)$ for a rather general type of the right hand side of Eq. (1). To this end we prove closure of the basis made up on squared Jost functions, which allows us to find a solution of the linear inhomogeneous equation associated with the perturbed NSE (Sec. II). Then in Sec. III we derive equations describing time dependence of soliton parameters and represent an evident expression for the first order addendum. In Sec. IV we consider particular examples related to the optical pulse evolution in nonlinear fibers.

II. BASIS OF THE LINEARIZED OPERATOR

A. Statement of the problem

A conventional way to construct any perturbation theory is to look for a solution in the form of an expansion with respect to a small parameter. Correspondingly we represent

$$q(x,t)=e^{i\mu\phi(x,t)}[q_0(x,t;\tau)+\mu q^{(1)}(x,t;\tau)] \ . \tag{4}$$

The leading order of the expansion has the form

$$q_0(x,t;\tau)=\rho\frac{1+\varepsilon^2\exp\{\nu(\tau)[x-vt-x_0(\tau)]\}}{1+\exp\{\nu(\tau)[x-vt-x_0(\tau)]\}} \tag{5}$$

in which the dependence on a slow time

$$\tau=\mu t \tag{6}$$

is introduced in an explicit form. Whenever we speak about the adiabatic approximation (5) we have to distinguish $\omega\sin(\vartheta/2)$ from $\nu(\tau)$; also the velocity of the soliton is equal to $-\omega\cos(\vartheta/2)$ only at $t=0$.

In contrast to the case of bright solitons, it turns out to be useful to introduce an additional varying phase $\phi(x,t)$, the role of which will be clear below. Naturally in order to satisfy boundary conditions (2) we have to require that

$$\phi(x,t)\to 0 \text{ at } |x|\to\infty \ . \tag{7}$$

Also, as far as a soliton dynamics is under consideration we take the initial condition for the phase as

$$\phi(x,t=0)=0 \ . \tag{8}$$

Here we should emphasize the difference of our statement of the problem from that of [18]. There are two possibilities to formulate the perturbation theory of a dark soliton. We will call them approaches of *varying* and *constant* phases.

The approach of varying phase, which was used in [18], implies a variation of the phase ϑ with slow time. In this case only the modulus of the solution is considered to be fixed at the infinity. Also in that case the relation among the parameters ν, v, and ϕ is the same as in the unperturbed dynamics. This approach does have a physical background, but it derives the solution from the class defined by (2).

In the present paper we concentrate on the *approach of the constant phase* [defined by (7) and (8)], which has to be implied whenever the boundary conditions, including the phase, are fixed. It is worth pointing out that the analysis of the linearized problem, provided below in this paper, evidently makes up the necessary basis to build up the regular perturbative expansion for the varying phases (which will be reported elsewhere).

Inserting (4) into (1) and holding terms of the first order of μ one gets the linear system

$$\mathcal{L}|Q^{(1)}\rangle=\sigma_3|P\rangle \ . \tag{9}$$

Here the operator $\mathcal{L}$ is given by

$$\mathcal{L}=i\frac{\partial}{\partial t}I-\left[\frac{\partial^2}{\partial x^2}+2\rho^2-4|q_s|^2\right]\sigma_3+2\begin{bmatrix}0 & \bar{q}_0\\ -q_0 & 0\end{bmatrix} , \tag{10}$$

with I the unit matrix, $\sigma_j (j=1,2,3)$ the Pauli matrices, and a ket vector

$$|Q^{(1)}\rangle=\text{col}(\bar{q}_1,q_1) \ . \tag{11}$$

Throughout the paper values with an overbar are used for corresponding conjugative ones. The right hand side of Eq. (9) is given by $|P\rangle=\text{col}(\bar{p}_1,p_1)$, where

$$p_1=\hat{R}-i\frac{\partial q_0}{\partial\tau}+(\phi_t-i\phi_{xx})q_0-2i\phi_x q_{0x} \ . \tag{12}$$

It is worth pointing out that we make a use of a matrix system containing the linearized equation (1) and its complex conjugation, which is natural for a general formalism [9].

B. Squared Jost functions

The unperturbed NSE is a compatibility condition for the two systems

$$\frac{\partial\Psi}{\partial x}=U(x,t;z)\Psi \ , \tag{13}$$

$$\frac{\partial\Psi}{\partial t}=V(x,t;z)\Psi \tag{14}$$

(the UV pair can be found, e.g., in [19]; for the sake of convenience we present it in Appendix A).

The monodromy matrix $T(z)$ bounds Jost functions $T_\pm(x;z)$ [19]

$$T_-(x;z)=T_+(x;z)T(z) \ , \tag{15}$$

which are solutions of Eq. (13) defined by the asymptotic expressions

$$T_+(x;z)\to\exp\left[-\frac{i\vartheta\sigma_3}{2}\right]E(x;z) \text{ at } \to\infty \ , \tag{16}$$

$$T_-(x;z)\to E(x;z) \text{ at } x\to-\infty \tag{17}$$

with

$$E(x;z)=\begin{bmatrix}1 & -i\omega/z\\ i\omega/z & 1\end{bmatrix}\exp\left[-\frac{ikx}{2}\sigma_3\right] \tag{18}$$

and $k(z)=\frac{1}{2}(z-\omega^2/z)$.

It is not difficult to verify that $\tilde{\Psi}_j=\mathrm{col}(\Psi^{(1j)2},\Psi^{(2j)2})$ solves the homogeneous equation

$$\mathcal{L}\tilde{\Psi}_j=0 \tag{19}$$

if $\Psi^{(ij)}$ are elements of a matrix satisfying the joint system (13) and (14).

Let us introduce squared Jost functions

$$|F_{\pm}^{(j)}(x;z)\rangle=\begin{bmatrix}[T_{\pm}^{(1j)}(x;z)]^2\\ [T_{\pm}^{(2j)}(x;z)]^2\end{bmatrix} . \tag{20}$$

Direct algebra allows one to ensure that in the one soliton case $|F_{\pm}^{(i)}(x;z)\rangle$ are eigenfunctions of the operator $\mathcal{L}$

$$\mathcal{L}|F_{\pm}^{(j)}(x;z)\rangle=\frac{(-1)^{1+j}}{4}k(z)\lambda(z)|F_{\pm}^{(j)}(x;z)\rangle \tag{21}$$

with $\lambda(z)=\frac{1}{2}(z+\omega^2/z)$.

Thus if we determine a space and prove closure of a set of functions $|F_{\pm}^{(j)}(x;z)\rangle$, then we will be able to obtain a Green's function of the operator $\mathcal{L}$. One of important properties of the squared Jost functions used in this way is that they link variations of scattering data with small variations of the potential. Namely, one has

$$\delta r(z)=\frac{1}{\Delta(z)T_{11}^2(z)}\int_{-\infty}^{\infty}dx\,\langle F_-^{(1)}(x;z)|\delta Q(x)\rangle , \tag{22}$$

$$\delta\tilde{r}(z)=\frac{1}{\Delta(z)T_{22}^2(z)}\int_{-\infty}^{\infty}dx\,\langle F_-^{(2)}(x;z)|\delta Q(x)\rangle , \tag{23}$$

$$\gamma_k\delta z_k=\frac{1}{\Delta^2(z_k)\dot{T}_{11}^2(z_k)}\int_{-\infty}^{\infty}dx\,\langle F_-^{(1)}(x;z_k)|\delta Q(x)\rangle , \tag{24}$$

$$\tilde{\gamma}_k\delta\tilde{z}_k=\frac{1}{\Delta^2(\tilde{z}_k)\dot{T}_{22}^2(\tilde{z}_k)}\int_{-\infty}^{\infty}dx\,\langle F_-^{(2)}(x;\tilde{z}_k)|\delta Q(x)\rangle , \tag{25}$$

$$\delta\gamma_k=\frac{1}{\dot{T}_{11}(z_k)}\left[\frac{\partial}{\partial z}\frac{1}{\Delta^2(z)\dot{T}_{11}(z)}\times\int_{-\infty}^{\infty}dx\,\langle F_-^{(1)}(x;z)|\delta Q(x)\rangle\right]_{z=z_k} , \tag{26}$$

$$\delta\tilde{\gamma}_k=\frac{-1}{\dot{T}_{22}(\tilde{z}_k)}\left[\frac{\partial}{\partial z}\frac{1}{\Delta^2(z)\dot{T}_{22}(z)}\times\int_{-\infty}^{\infty}dx\,\langle F_-^{(2)}(x;z)|\delta Q(x)\rangle\right]_{z=\tilde{z}_k} . \tag{27}$$

Here we use conventional designations

$$r(z)=\frac{T_{21}(z)}{T_{11}(z)},\quad \tilde{r}(z)=\frac{T_{12}(x)}{T_{22}(z)} ; \tag{28}$$

$$\gamma_k=\gamma(z_k),\quad \tilde{\gamma}_k=\tilde{\gamma}(\tilde{z}_k),\qquad \gamma(z)=\frac{T_{21}(z)}{\Delta(z)\dot{T}_{11}(z)},\quad \tilde{\gamma}(z)=\frac{T_{12}(z)}{\Delta(z)\dot{T}_{22}(z)} ; \tag{29}$$

$$\delta\gamma_k=\delta\gamma(z_k)+\dot{\gamma}(z_k)\delta z_k,\qquad \delta\tilde{\gamma}_k=\delta\tilde{\gamma}(\tilde{z}_k)+\dot{\tilde{\gamma}}(\tilde{z}_k) ; \tag{30}$$

$$\Delta(z)=1-\frac{\omega^2}{z^2} \tag{31}$$

where z_k and $\tilde{z}_k$ are zeros of $T_{11}(z)$ and $T_{22}(z)$ in the upper and the lower z half plane, respectively; $T_{11}(z_k)=T_{22}(\tilde{z}_k)=0$, overdots denote the derivative with respect to the argument, and $\delta f(z_k)$ means $[\delta f(z)]_{z=z_k}$. Representations (22)–(27) are obtained from the well known formula [19]

$$\delta T(z)=\int dx\,T_+^{-1}(x;z)\delta U(x;z)T_-(x;z) , \tag{32}$$

where

$$\delta U(x;z)=\begin{bmatrix}0 & \delta\bar{q}\\ \delta q & 0\end{bmatrix} \tag{33}$$

describes small variations of the field $q(x,t)$.

C. Orthogonality

As is customary, in order to state orthogonality relations we notice that the squared Jost functions solve an equation

$$\Upsilon_{\pm}^{(j)}|F_{\pm}^{(j)}(x,z)\rangle=-i\lambda(z)|F_{\pm}^{(j)}(x,z)\rangle \tag{34}$$

with the integro-differential operator

$$\Upsilon_{\pm}^{(j)}=\begin{bmatrix}\frac{\partial}{\partial x}-2\bar{q}(x)\int_{g_{\pm}^j}^{x}d\xi\, q(\xi) & 2\bar{q}(x)\int_{g_{\pm}^j}^{x}d\xi\,\bar{q}(\xi)\\ -2q(x)\int_{g_{\pm}^j}^{x}d\xi\, q(\xi) & -\frac{\partial}{\partial x}+2q(x)\int_{g_{\pm}^j}^{x}d\xi\,\bar{q}(\xi)\end{bmatrix} , \tag{35}$$

where $g_{\pm}^j$ are constants which must be found from the relation

$$T_{\pm}^{(1j)}T_{\pm}^{(2j)}=\int_{g_{\pm}^j}^{x}d\xi[\bar{q}(\xi)F_{\pm}^{(2j)}+q(\xi)F_{\pm}^{(1j)}] \tag{36}$$

to satisfy the limiting transitions $x\to\pm\infty$. Then there exists an equality

$$\langle F_-^{(l)}(x;z')|\Upsilon_+^{(j)}|F_+^{(j)}(x;z)\rangle+\langle F_+^{(j)}(x;z)|\Upsilon_-^{(l)}|F_-^{(l)}(x;z')\rangle =-i[\lambda(z)-\lambda(z')]\langle F_-^{(l)}(x;z')|F_+^{(j)}(x;z)\rangle \tag{37}$$

(some general properties of the respective vector algebra are outlined in Appendix B). On the other hand, direct algebra allows one to represent the left hand side of (37) in the form

$$-\frac{\partial}{\partial x}[\langle T_{+}^{(j)}(x;z)|T_{-}^{(l)}(x;z')\rangle]^2 . \tag{38}$$

Thus, using asymptotic expressions (16) and properties of Dirac's delta function one finds the orthogonality conditions

$$\int_{-\infty}^{\infty} dx\,\langle F_{-}^{(l)}(x;z')|F_{+}^{(j)}(x;z)\rangle = 4\pi(1-\delta_{lj})\Delta(z)T_{ll}^{2}(z)\delta(z-z') \tag{39}$$

if $\mathrm{Im}z=0, |z|>\omega$. It is worth pointing out here that sensitivity of the orthogonality relations to the location of a spectral parameter is predictable due to an analogy with the well-known behavior of the eigenfunctions of the operator $i\sigma_3 d/dx+(\omega/2)\sigma_2$ [19].

D. Expansion of the unity

In order to express variations of the field through variations of the scattering data we will use the formalism of the Reimann problem associated with the unperturbed NSE. Omitting details, which can be found, say, in [19], we briefly outline the final statement.

Let us introduce matrices $\Psi_{\pm}(x;z)$

$$\Psi_{\pm}(x;z)=T_{\pm}(x;z)\exp\left[\frac{ikx}{2}\sigma_3\right] . \tag{40}$$

Then

$$\Psi_{-}(x;z)=\Psi_{+}(x;z)S(x;z) , \tag{41}$$

where the matrix $S(x;z)$ is expressed through the monodromy matrix

$$S(x;z)=\exp\left[-\frac{ikx}{2}\sigma_3\right]T(z)\exp\left[\frac{ikx}{2}\sigma_3\right] . \tag{42}$$

Designating by single upper indices columns of corresponding matrices we make up new ones

$$H_{\pm}=\left[\Psi_{\mp}^{(1)}\Psi_{\pm}^{(2)}\right] . \tag{43}$$

It is not difficult to verify (see [19]) that H_+ and H_- are analytical in the upper and lower half planes of z, respectively.

The Reiman problem can be formulated in terms of matrices

$$H(x;z)=H_{+}(x;z), G(x;z)=H_{-}^{\dagger}(x;z) , \tag{44}$$

where a conjugative matrix is defined by the rule (B4). Direct algebra yields

$$G(x;z)H(x;z)=\Delta(z)\begin{bmatrix} 1 & -S_{12}(x;z) \\ S_{21}(x;z) & 1 \end{bmatrix} \tag{45}$$

at $\mathrm{Im}z=0$.

Varying the linear spectral problem (13) and using the relation $H^{-1}H_xH^{-1}=-(H^{-1})_x$, valid for any nondegenerate matrix H, one gets the representation

$$\delta U=(\delta HH^{-1})_x-[U,\delta HH^{-1}] = -(G^{-1}\delta G)_x+[U,G^{-1}\delta G] . \tag{46}$$

Using analytical properties of matrices H and G this expression may be rewritten in the form

$$\delta U=\Lambda_x^{(-)}-[U,\Lambda^{(-)}] . \tag{47}$$

Hereafter we use designations

$$\Lambda^{(\pm)}(x;z)=\delta HH^{-1}\theta(\mathrm{Im}z)\pm G^{-1}\delta G\theta(-\mathrm{Im}z) \tag{48}$$

[with $\theta(\ )$ the Heaviside step function] for appropriate combinations analytical in the whole z plane. Our aim now is to evaluate $\Lambda^{(-)}$. To this end we recall Eq. (45), which after variation yields on the real axis

$$\Lambda^{(+)}=\frac{1}{2}\Delta(z)G^{-1}\begin{bmatrix} 0 & -\delta S_{12} \\ \delta S_{21} & 0 \end{bmatrix}H^{-1} . \tag{49}$$

It follows from (46) and (48) that $\Lambda^{(+)}$ solves the equation

$$\Lambda_x^{(+)}=[U,\Lambda^{(+)}] . \tag{50}$$

Now we employ properties of the matrix H as follows: (i) It is analytical in the upper half plane of z. (ii) It is degenerate in points z_k, so that elements of the inverse matrix have simple poles. (iii) Its behavior in points $\pm\omega$ is different in a generic case and in a pure soliton case. Then, using the Cauchy formula we derive (by closing the contour in the upper half plane)

$$\delta H(x;z)H^{-1}(x;z)\theta(\mathrm{Im}z) = \frac{1}{2\pi i}⨍\frac{d\xi}{\xi-z}\delta H(x;\xi)H^{-1}(x;\xi)\theta(\mathrm{Im}\xi) + \sum_k\frac{1}{z-z_k}\mathrm{Res}_{z_k}\delta HH^{-1}-(I_{+}+I_{-}) . \tag{51}$$

By analogy for $G^{-1}\delta G$ we have

$$G^{-1}(x;z)\delta G(x;z)\theta(-\mathrm{Im}z) = \frac{1}{2\pi i}⨍\frac{d\xi}{\xi-z}G^{-1}(x;\xi)\delta G(x;\xi)\theta(-\mathrm{Im}\xi) + \sum_k\frac{1}{z-\tilde{z}_k}\mathrm{Res}_{\tilde{z}_k}G^{-1}G-(\tilde{I}_{+}+\tilde{I}_{-}) . \tag{52}$$

In formulas (51) and (52) we use the designations

$$⨍=\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\omega-\varepsilon}+\int_{-\omega+\varepsilon}^{\omega-\varepsilon}+\int_{\omega+\varepsilon}^{\infty}\right] , \tag{53}$$

$$I_{\pm}=\frac{1}{2\pi i}\int_{\gamma_{\pm}}\frac{d\xi}{\xi-z}\delta H(x;\xi)H^{-1}(x;\xi) , \tag{54}$$

$$\tilde{I}_{\pm}=\frac{1}{2\pi i}\int_{\tilde{\gamma}_{\pm}}\frac{d\xi}{\xi-z}G^{-1}(x;\xi)\delta G(x;\xi) . \tag{55}$$

Contours $\gamma_{\pm}$ and $\tilde{\gamma}_{\pm}$ are shown in Fig. 1 and summation in (51) and (52) is carried over all poles z_k and $\tilde{z}_k$ num-

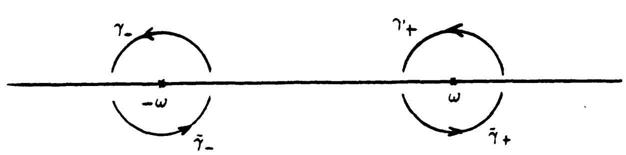

FIG. 1. Contours $\gamma_\pm$ and $\tilde{\gamma}_\pm$ around the edges of the continuum spectrum.

bered correspondingly by a subindex k. Thus for $\Lambda^{(-)}$ we have a representation

$$\begin{aligned}\Lambda^{(-)}(x;z)=&\frac{1}{2\pi i}\oint\frac{d\xi}{\xi-z}\Lambda^{(+)}(x;\xi)\\&+\sum_k\frac{1}{z-z_k}\mathrm{Res}_{z_k}\delta HH^{-1}\\&+\sum_k\frac{1}{z-\bar{z}_k}\mathrm{Res}_{\bar{z}_k}G^{-1}\delta G\\&+\tilde{I}_+-I_++\tilde{I}_--I_-\ .\end{aligned}\tag{56}$$

In the case of general situation $\tilde{I}_\pm=I_\pm=0$, which is a consequence of the independence of the columns $T_-^{(1)}(x;+\omega)$ and $T_+^{(2)}(x;\pm\omega)$ [and $T_-^{(2)}(x;\pm\omega)$ and $T_+^{(1)}(x;\pm\omega)$] [19]. However, in the pure soliton case the integrals mentioned give a nonzero contribution. Namely we have (Appendix C)

$$I_\pm-\tilde{I}_\pm=\frac{\alpha_\pm}{2(\pm\omega-z)S_{11}(\pm\omega)}I\ ,\tag{57}$$

where $\alpha_\pm$ are residues of δT_{11} in points $\pm\omega$. Thus there exist two properties

$$\frac{\partial}{\partial x}(I_\pm-\tilde{I}_\pm),\quad [B,I_\pm-\tilde{I}_\pm]=0\ ,\tag{58}$$

where B is an arbitrary matrix.

Inserting the expression (56) in (47), using (45), (48), and properties (51) and (58), one finds

$$\delta U=[\sigma_3,M]\tag{59}$$

where

$$\begin{aligned}M=&\frac{1}{4}\oint d\xi\Lambda^{(+)}(x;\xi)\\&+\frac{i}{4}\sum_k(\mathrm{Res}_{z_k}\delta HH^{-1}-\mathrm{Res}_{\bar{z}_k}G^{-1}\delta G)\ .\end{aligned}\tag{60}$$

In order to calculate M we notice that $\Lambda^{(+)}$ can be represented in the form

$$2\Lambda^{(+)}=\Lambda_0+\Lambda_1+\tilde{\Lambda}_0+\tilde{\Lambda}_1\tag{61}$$

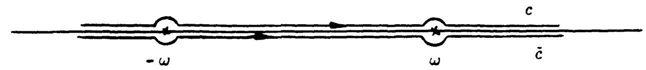

FIG. 2. Contours c and $\tilde{c}$.

where

$$\Lambda_0=|T_+^{(2)}\rangle\frac{\delta r}{\Delta}\langle T_+^{(2)}|,\quad \Lambda_1=|T_-^{(1)}\rangle\frac{\delta T_{11}}{\Delta T_{11}^2}\langle T_+^{(2)}|\ ,\tag{62}$$

$$\tilde{\Lambda}_0=|T_+^{(1)}\rangle\frac{\delta\tilde{r}}{\Delta}\langle T_+^{(1)}|,\quad \tilde{\Lambda}_1=|T_+^{(1)}\rangle\frac{\delta T_{22}}{\Delta T_{22}^2}\langle T_-^{(2)}|\ .\tag{63}$$

Correspondingly, the integral in (60) can be splitted into four parts. Since Λ_1 and $\tilde{\Lambda}_1$ can be analytically continued to the upper and lower z half planes, one can write

$$\oint d\xi\Lambda_1=\sum_k\mathrm{Res}_{z_k}\Lambda_1+\left[\int_{\gamma+}+\int_{\gamma-}\right]d\xi\Lambda_1\ ,\tag{64}$$

$$\oint d\xi\tilde{\Lambda}_1=\sum_k\mathrm{Res}_{\bar{z}_k}\tilde{\Lambda}_1+\left[\int_{\tilde{\gamma}+}+\int_{\tilde{\gamma}-}\right]d\xi\tilde{\Lambda}_1\ .\tag{65}$$

Two other parts having Λ_0 and $\tilde{\Lambda}_0$ in integrands are represented as

$$\oint d\xi\Lambda_0=\int_c d\xi\Lambda_0+\left[\int_{\gamma+}+\int_{\gamma-}\right]d\xi\Lambda_0\ ,\tag{66}$$

$$\oint d\xi\tilde{\Lambda}_0=\int_{\tilde{c}} d\xi\tilde{\Lambda}_0+\left[\int_{\tilde{\gamma}+}+\int_{\tilde{\gamma}-}\right]d\xi\tilde{\Lambda}_0\tag{67}$$

(contours c and $\tilde{c}$ are sketched in Fig. 2). As above, integrals over $\gamma_\pm$ and $\tilde{\gamma}_\pm$ equal zero in the case of a general situation. In the pure soliton case (see Appendix C)

$$\int_{\gamma\pm}d\xi(\Lambda_0+\Lambda_1)-\int_{\tilde{\gamma}\pm}d\xi(\tilde{\Lambda}_0+\tilde{\Lambda}_1)=-i\frac{\pi\alpha_\pm}{T_{11}(\pm\omega)}I\tag{68}$$

the residues appearing after the substitution of (64) and (65) into (60) are evaluated (Appendix D):

$$\begin{aligned}&\mathrm{Res}_{z_k}(\delta HH^{-1}+\Lambda_1)\\&\quad=-\delta\gamma_k|T_+^{(2)}(z_k)\rangle\langle T_+^{(2)}(z_k)|\\&\quad\quad-\gamma_k\delta z_k\left[\frac{\partial}{\partial z}|T_+^{(2)}(z)\rangle\langle T_+^{(2)}(z)|\right]_{z=z_k}\ ,\end{aligned}\tag{69}$$

$$\begin{aligned}&\mathrm{Res}_{\bar{z}_k}(G^{-1}\delta G+\tilde{\Lambda}_1)\\&\quad=-\delta\bar{\gamma}_k|T_+^{(1)}(\bar{z}_k)\rangle\langle T_+^{(1)}(\bar{z}_k)|\\&\quad\quad-\bar{\gamma}_k\delta\bar{z}_k\left[\frac{\partial}{\partial z}|T_+^{(1)}(z)\rangle\langle T_+^{(1)}(z)|\right]_{z=\bar{z}_k}\ .\end{aligned}\tag{70}$$

Using (70) and (71), the variation δU, given by (59) and (60), can be represented in the form

$$\begin{aligned}|\delta U\rangle=&\frac{1}{4\pi}\int_c\frac{d\xi}{\Delta(\xi)}\delta r(\xi)|F_+^{(2)}(x;\xi)\rangle+\frac{1}{4\pi}\int_{\tilde{c}}\frac{d\xi}{\Delta(\xi)}\delta\tilde{r}(\xi)|F_+^{(1)}(x;\xi)\rangle\\&+\frac{1}{2i}\sum_k[\delta\gamma_k|F_+^{(2)}(x;z_k)\rangle+\gamma_k\delta z_k|\dot{F}_+^{(2)}(x,z_k)\rangle]-\frac{1}{2i}\sum_k[\delta\bar{\gamma}_k|F_+^{(1)}(x;\bar{z}_k)\rangle-\bar{\gamma}_k\delta\bar{z}_k|\dot{F}_+^{(2)}(x,\bar{z}_k)\rangle]\ .\end{aligned}\tag{71}$$

The last step is to insert (22)–(27) into this formula. As a result we have

$$\delta(x-x')I=\frac{1}{4\pi}\int_c d\xi|F_+^{(2)}(x;\xi)\rangle\frac{1}{\Delta^2(\xi)T_{11}^2(\xi)}\langle F_-^{(1)}(x',\xi)|-\frac{1}{4\pi}\int_{\tilde{c}} d\xi|F_+^{(1)}(x;\xi)\rangle\frac{1}{\Delta^2(\xi)T_{22}^2(\xi)}\langle F_-^{(2)}(x',\xi)|$$
$$+\frac{1}{2i}\sum_k\frac{1}{\dot{T}_{11}(z_k)}\left[\frac{\partial}{\partial z}|F_+^{(2)}(x;z)\rangle\frac{1}{\Delta^2(z)\dot{T}_{11}(z)}\langle F_-^{(1)}(x',z)|\right]_{z=z_k}$$
$$+\frac{1}{2i}\sum_k\frac{1}{\dot{T}_{22}(\tilde{z}_k)}\left[\frac{\partial}{\partial z}|F_+^{(1)}(x;z)\rangle\frac{1}{\Delta^2(z)\dot{T}_{22}(z)}\langle F_-^{(2)}(x',z)|\right]_{z=\tilde{z}_k} . \quad (72)$$

The expansion of the unity (72) allows a more compact representation, which reads

$$\delta(x-x')I=\frac{1}{4\pi}\int_\Gamma\Omega(x,x';z)dz-\frac{1}{4\pi}\int_{\tilde{\Gamma}}\tilde{\Omega}(x,x';z)dz , \quad (73)$$

where

$$\Omega(x,x';z)=|F_+^{(2)}(x;z)\rangle\frac{1}{\Delta^2(z)T_{11}^2(z)}\langle F_-^{(1)}(x';z)| , \quad (74)$$

$$\tilde{\Omega}(x,x';z)=|F_+^{(1)}(x;z)\rangle\frac{1}{\Delta^2(z)T_{22}^2(z)}\langle F_-^{(2)}(x';z)| , \quad (75)$$

and contour $\Gamma(\tilde{\Gamma})$ goes over (under) all zeros $z_k(\tilde{z}_k)$ in the upper (lower) z half plane. In particular, for the case under consideration Γ and $\tilde{\Gamma}$ are outside the circle $|z|=\omega$ (Fig. 3).

III. PERTURBED DYNAMICS OF A DARK SOLITON

A. General formulas

Now we have all that is necessary to study the dynamics of a dark soliton affected by perturbations. The perturbation $\sigma_3|P(t,x)\rangle$ is represented in the form (valid for any ket vector)

$$\sigma_3|P(t,x)\rangle=\frac{1}{4\pi}\int_\Gamma dz\frac{p(t,z)}{\Delta^2(z)T_{11}^2(z)}|F_+^{(2)}(t,x;z)\rangle$$
$$-\frac{1}{4\pi}\int_{\tilde{\Gamma}} dz\frac{\tilde{p}(t,z)}{\Delta^2(z)T_{22}^2(z)}|F_+^{(1)}(t,x;z)\rangle . \quad (76)$$

The coefficients of this expansion are determined by

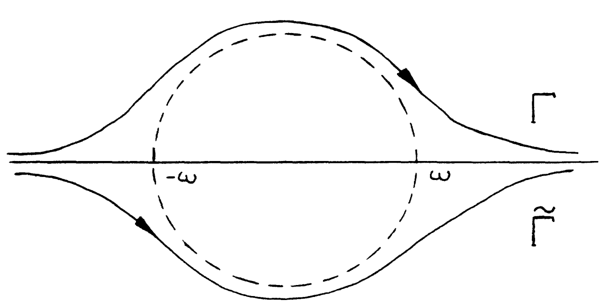

FIG. 3. Contours Γ and $\tilde{\Gamma}$.

$$p(t,z)=\int_{-\infty}^{\infty}dx\langle F_-^{(1)}(t,x;z)|\sigma_3|P(t,x)\rangle , \quad (77)$$

$$\tilde{p}(t,z)=\int_{-\infty}^{\infty}dx\langle F_-^{(2)}(t,x;z)|\sigma_3|P(t,x)\rangle . \quad (78)$$

Here we have used the orthogonality condition (39).

Correspondingly the solution of the linear problem we are looking for is represented in the form

$$|\delta Q(t,x)\rangle=\frac{1}{4\pi}\int_\Gamma dz\frac{y(t,z)}{\Delta^2(z)T_{11}^2(z)}|F_+^{(2)}(t,x;z)\rangle$$
$$-\frac{1}{4\pi}\int_{\tilde{\Gamma}} dz\frac{\tilde{y}(t,z)}{\Delta^2(z)T_{22}^2(z)}|F_+^{(1)}(t,x;z)\rangle . \quad (79)$$

Then applying the operator $\mathcal{L}$ to both sides of (79), equating the result to (76), and taking into account zero boundary conditions we compute

$$y(t,z)=-i\exp[-ik(z)\lambda(z)t]\times\int_0^t dt'p(t';z)\exp[ik(z)\lambda(z)t'] , \quad (80)$$

$$\tilde{y}(t,z)=-i\exp[ik(z)\lambda(z)t]\times\int_0^t dt'\tilde{p}(t';z)\exp[-ik(z)\lambda(z)t'] . \quad (81)$$

Formulas (79)–(81) completely define the first order addendum in the dynamics of the perturbed dark soliton.

After the insertion of (80), (81), (77), and (78) into (79), formula (81) yields the first order addendum in the perturbed dynamics of the dark soliton.

B. Excluding secular terms

Let us analyze the results obtained concentrating on one-soliton dynamics. After the insertion of (80) into (79) and changing the order of integration one finds that the integral with respect to z can be transformed to the sum of the integral over the real axis, the contribution of poles inside the circle $|z|=\omega$, and the contribution of the edges of the continuum spectrum $z=\pm\omega$ [such a representation can be directly obtained with the help of (73)]. We treat these parts separately.

Starting with the discrete spectrum, we notice that in the points z_1 and $\tilde{z}_1$ (the suffix k has been replaced by 1) there are second order poles of the respective integrands in (79). This results in the temporal growth of the amplitude of the solution $|\delta Q\rangle$. It is a secular growth which can be excluded by slowly varying parameters of the soliton. Bearing in mind that the order of the poles is two, we conclude that the condition for the corresponding contributions to be equal zero are

$$p(t,z_1)=\bar{p}(t,\bar{z}_1)=0 \tag{82}$$

and

$$\left.\frac{\partial p(t,z)}{\partial z}\right|_{z_1}=\left.\frac{\partial \bar{p}(t,z)}{\partial z}\right|_{\bar{z}_1}=0\,. \tag{83}$$

Since there exists an involution property $T_-^{(2)}(x,z)=\sigma_1 T_-^{(1)}(x,\bar{z})$ [19], it follows from the (77) and (78) that in fact one has to deal with either Eq. (82) and (83). For the sake of definiteness we will consider the pole z_1.

Passing to the contribution of the points $z=\pm\omega$, we take into account that the expansion is made about the one-soliton solution for which $T_{11}(\pm\omega)=\pm 1$. Hence, bearing in mind the involution properties (A9), one concludes that after a change of variables $z\to\omega^2/z$ in the second integral in (79), both terms can be combined. By this way it is found that the points $\pm\omega$ are first order poles of the integrands in (79). In a generic case $y(t,z)$ also may display secular growth because of integration over t. In order to prevent it one has to require that

$$p(t,\pm\omega)=0\,. \tag{84}$$

As above, due to the symmetry $T_+^{(1)}(x,\pm\omega)=\pm i T_+^{(2)}(x,\pm\omega)$ [19], (84) reduces to one equation, say for ω.

Let us insert into (82)–(84) the evident form of the one-soliton squared Jost function $\langle F_-^{(1)}|$ (see, e.g., [20]),

$$\langle F_-^{(1)}|=\left[\frac{\omega}{z}+\bar{\varepsilon}\right]^{-2} \times e^{-ikx}\left[\frac{\omega^2}{z^2}\left[\frac{\omega}{z}+\bar{\varepsilon}\frac{q_s}{\rho}\right]^2,\left[\frac{\omega}{z}\frac{\bar{q}_s}{\rho}+\bar{\varepsilon}\right]^2\right] \tag{85}$$

with q_s given by (3). As a result the equations at hand are reduced to

$$\int_{-\infty}^{\infty}\frac{dx}{\cosh^2\frac{\Theta}{2}}\text{Im}\bar{\varepsilon}p_1=0\,, \tag{86}$$

$$-\cos\frac{\vartheta}{2}\int_{-\infty}^{\infty}dx\left[\frac{\Theta/2}{\cosh^2\frac{\Theta}{2}}+\tanh\frac{\Theta}{2}\right]\text{Im}\bar{\varepsilon}p_1 +\sin\frac{\vartheta}{2}\int_{-\infty}^{\infty}dx\,\text{Re}\bar{\varepsilon}p_1=0\,, \tag{87}$$

$$\cos\frac{\vartheta}{2}\int_{-\infty}^{\infty}dx\,\text{Im}\bar{\varepsilon}p_1-\sin\frac{\vartheta}{2}\int_{-\infty}^{\infty}dx\tanh\frac{\Theta}{2}\text{Re}\bar{\varepsilon}p_1=0\,. \tag{88}$$

$\Theta=\nu(\tau)\zeta$ and $\zeta=x-vt-x_0(\tau)$, respectively. To derive (87) we took into account the relation (88).

Despite the fact that the unperturbed soliton is determined by two parameters [say $\lambda(z_1)$ and γ_k], we obtained three conditions. This is a peculiarity of the dark soliton dynamics. In other words, slow variations of only two parameters of the soliton cannot prevent the secular growth. To make this statement more evident we note that the term $i\bar{\varepsilon}\partial q_0/\partial\tau$ from (12) is real and hence its contribution to the integral (86) gives zero at any time dependence of the parameters $\nu(\tau)$ and $x_0(\tau)$. This means that if the perturbation $\hat{R}$ contains a part giving a nonzero contribution in (86), the respective secular growth cannot be compensated for by variations of the soliton parameters. It is the reason why the phase $\phi(x,t)$ is introduced in the representation (4).

C. On definition of phase

Now we consider the phase contribution in more detail. To this end we write Eq. (86) in the explicit form, taking into account the evident expression (12) for p_1,

$$\int_{-\infty}^{\infty}\frac{dx}{\cosh^2(\Theta/2)}\left[-\cos\frac{\vartheta}{2}\phi_{xx}+\sin\left[\frac{\vartheta}{2}\right]\phi_t\tanh\frac{\Theta}{2} +\frac{1}{\rho}\text{Im}\bar{\varepsilon}\hat{R}\right]=0\,. \tag{89}$$

To all appearances the choice of the phase is not unique. In particular, one could require the integrand in (89) to be zero. It is reasonable, however, to look for the expression for the phase coordinated with that obtained by another approach following from the conservation law for the momentum

$$P=\frac{1}{2i}\int_{-\infty}^{\infty}dx(q_x\bar{q}-q\,\bar{q}_x)\,. \tag{90}$$

To this end we integrate the term proportional to ϕ_{xx} in (89) by parts and make a transformation to new variables: $(x,t)\to(\Theta,t)$. As a result we obtain

$$\int_{-\infty}^{\infty}d\Theta\left[\frac{\sinh(\Theta/2)}{\cosh^3(\Theta/2)}\left[v\sin\frac{\vartheta}{2}-\nu\cos\frac{\vartheta}{2}\right]\nu\phi_\Theta +\sin\frac{\vartheta}{2}\phi_t\tanh\frac{\Theta}{2}+\frac{1}{\rho}\text{Im}\bar{\varepsilon}\hat{R}\right]=0 \tag{91}$$

[hereafter ϕ is considered as a function of (Θ,t)]. Noting that only the odd (with respect to Θ) part of the perturbation yields a contribution to the integral we introduce the designation

$$F(\Theta,t)=-\frac{1}{2}\coth\left[\frac{\Theta}{2}\right]\text{Im}\bar{\varepsilon}[\hat{R}(\Theta,t)+\hat{R}(-\Theta,t)]\,. \tag{92}$$

Then the phase subject to the conditions (7) and (8) can be found in the form of an odd function on Θ. Namely, we have

$$\phi=\frac{1}{\rho\sin(\vartheta/2)}\int_0^t dt' F(\Theta,t') \tag{93}$$

(note that in this case ϕ_Θ is an even function and its respective integral is equal to zero).

Direct calculation of $\partial P/\partial t$ allows one to verify that the expression (93) for the phase is obtained also from the modified conservation law for the momentum

$$\frac{\partial P}{\partial t}=-\mu\int_{-\infty}^{\infty}dx(\hat{R}\bar{q}_x-\bar{\hat{R}}q_x) . \tag{94}$$

However, in the way just described one meets the following problem. Unless $\hat{R}(\Theta=0,t)=0$, the phase diverges at $\Theta=0$. The nature of this phenomenon becomes clear from the original perturbed equation (1). Indeed, after substitution of the adiabatic form (5) into (1), one easily finds that the imaginary part of the left hand side multiplied by $\bar{\varepsilon}$ equals zero at $\Theta=0$. The factor $\mathrm{Im}\bar{\varepsilon}q_0$ at ϕ_t also is equal to zero. Thus, if $\mathrm{Im}\bar{\varepsilon}\hat{R}\neq 0$ at $\Theta=0$, the phase does not exist unless $\phi_t=\infty$. It is a situation when the "adiabatic" term requires global renormalization (and even the term "adiabatic" no longer has the conventional sense). Moreover, taking into account that the phase is an odd function of Θ we have to require that

$$\mathrm{Im}\bar{\varepsilon}\hat{R}''_{+}(\Theta,t)\sim\Theta^2 \text{ at } \Theta\to 0 . \tag{95}$$

In what follows we restrict our consideration only to this case.

D. Equations of the adiabatic approximation

Having introduced the phase we can rewrite Eqs. (87) and (88) in the form conventional for the adiabatic approximation,

$$\frac{dx_0}{dt}=\mu\int_{-\infty}^{\infty}d\Theta\left\{\frac{1}{v_0\nu}R'_{+}(\Theta,t)-\frac{v_0}{v_0^2\nu}\left[\frac{\Theta/2}{\cosh^2(\Theta/2)}+\tanh\frac{\Theta}{2}\right]R''_{-}(\Theta,t)\right.$$
$$\left.-\frac{\omega^2\nu}{v_0^3}\int_0^t dt'\frac{R''_{+}(\Theta,t')}{\sinh\Theta}\left[\left[1-4\frac{v_0^2}{\omega^2}\right]\tanh\frac{\Theta}{2}+\frac{v_0^2}{\omega^2}\Theta\left[1-\frac{3}{2}\frac{1}{\cosh^2(\Theta/2)}\right]\right]\right\}, \tag{96}$$

$$\frac{d\nu}{dt}=-\mu\frac{\nu}{\nu_0}\int_{-\infty}^{\infty}d\Theta\left[\tanh\left[\frac{\Theta}{2}\right]R'_{-}(\Theta,t)+\frac{v_0}{\nu_0}R''_{+}(\Theta,t)\right] . \tag{97}$$

Here we introduce the designations

$$R'_{\pm}(\Theta,t)=\frac{1}{2}\mathrm{Re}[\bar{\varepsilon}\hat{R}(\Theta,t)\pm\varepsilon\bar{\hat{R}}(-\Theta,t)] , \tag{98}$$

$$R''_{\pm}(\Theta,t)=\frac{1}{2}\mathrm{Im}[\bar{\varepsilon}\hat{R}(\Theta,t)\mp\varepsilon\bar{\hat{R}}(-\Theta,t)] . \tag{99}$$

Also

$$\nu_0=\omega\sin\frac{\vartheta}{2} \tag{100}$$

is the initial value of $\nu(t)$:$\nu(t=0)=\nu_0$; a conventional designation $v_0=-\omega\cos\vartheta/2$ is used.

It is worthwhile to mention that the equation for ν can be obtained directly from the first conservation law

$$\frac{dN}{dt}=2\mu\int_{-\infty}^{\infty}dx\,\mathrm{Im}(\bar{q}\hat{R}) , \tag{101}$$

where

$$N=\int_{-\infty}^{\infty}dx(|q|^2-\rho^2) . \tag{102}$$

Formulas (96) and (97) allow the general treatment in the case $\hat{R}\equiv\hat{R}(\Theta)$. The equation for ν is singled out. It takes the form

$$\frac{d\nu}{dt}=\mu A\nu , \tag{103}$$

where A is a constant evidently defined by (97). In other words,

$$\nu=\nu_0\exp(\mu At) . \tag{104}$$

IV. EXAMPLES

Let us consider now some particular examples of the perturbed dynamics of a dark soliton.

A. Dark soliton under random perturbation

Evolution of picosecond optical pulses in the regime of normal group velocity dispersion in a single-mode fiber is governed by Eq. (1) with $\mu=0$ [16]. In order to describe the effect of the fluctuation of linear and nonlinear susceptibilities, or the derivations of the core radius along the fiber axis, one has to write down the right hand side of (1) in the form

$$f(x,t)R[q]\to\mu[\epsilon(t)q+\chi(t)|q|^2q] , \tag{105}$$

where $\epsilon(t)$ and $\chi(t)$ are arbitrary (in particular, random) functions on time.

Though the perturbation (105) does not decay with $|\Theta|$, it is not difficult to verify that the substitution

$$q(x,t)=\exp\left[i\mu\int_0^t dt'[\epsilon(t')+\rho^2\chi(t')]\right]u(x,t) \tag{106}$$

reduces the problem at hand to the equation

$$iu_t+u_{xx}+(\rho^2-|u|^2)u=\mu\chi(t)(\rho^2-|u|^2)u \tag{107}$$

with the perturbation satisfying all necessary requirements.

Considering initial conditions for $q(x,t)$ being a dark soliton, $u(x,t)$ can be treated within the framework of the theory developed above. For the perturbation of the form $\chi(t)(\rho^2-|u|^2)u$ we immediately obtain

$$\phi(x,t)\equiv 0, \quad \nu=\text{const}, \quad x_0(t)=x_0(0)-\mu\frac{v}{4}\int_0^t dt'\chi(t') . \tag{108}$$

Comparing the outcome with the one known for a bright soliton case [22] one finds an essential difference. The nonlinear term with $\chi(t)$ leads to fluctuations of the dark soliton velocity, while it results only in the phase fluctuations of a bright soliton.

B. Evolution of the black soliton

In the case when the perturbation has the structure as follows:

$$\varepsilon\hat{R}(\Theta)=iR''_+(\Theta)+R'_-(\Theta) , \tag{109}$$

the phase is obtained to be

$$\phi=-\frac{t}{\rho\sin(\vartheta/2)}\coth\left[\frac{\Theta}{2}\right]R''_+(\Theta) . \tag{110}$$

A particular example of such a situation is the propagation of femtosecond optical pulses in nonlinear single-mode optical fibers [16].

If also $\vartheta=\pi$, we are dealing with the so-called black soliton (the terminology accepted in optics is used). In this case $v_0=0, \nu_0=\omega$, and Eqs. (96) and (97) are simplified drastically

$$\frac{dx_0}{dt}=\mu\frac{2}{\omega\nu}\int_{-\infty}^{\infty}d\Theta R'_+(\Theta,t) -\mu\frac{\nu}{2\nu_0^2}\int_0^t dt'\int_{-\infty}^{\infty}\frac{d\Theta}{\cosh^2(\Theta/2)}R''_+(\Theta,t') , \tag{111}$$

$$\frac{d\nu}{dt}=-\mu\frac{\nu}{\nu_0}\int_{-\infty}^{\infty}d\Theta\tanh\left[\frac{\Theta}{2}\right]R'_-(\Theta,t) . \tag{112}$$

As it could be expected, the velocity of the black soliton is determined only by a component of the perturbation having the parity opposite to that of the soliton.

V. CONCLUSION

To conclude, we have obtained the first order addendum to the dark soliton caused by perturbation. In a generic case the dynamics of a dark soliton has peculiarities when comparing it to the corresponding situation in a bright soliton case. It is the appearance of the rapid phase modulation of the leading order term in the perturbative expansion. This is unlike the conventional adiabatic approach in the perturbation theory of solitons.

Another peculiarity of the dark soliton dynamics is connected with the fact that a phase modulation leads to the creation of new solitons [20]. The last process is thresholdless and hence should be taken into account.

The consideration has been restricted to the case of fixed phases. Meanwhile, as far as the Green's function of the linearized problem has been defined, it is not difficult to extend the theory for the case of varying phases, when $\phi(x,t)\equiv\phi(t)$.

The above problems will be considered elsewhere.

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APPENDIX A

The operators U and V are given by [19]

$$U(x,t;z)=\begin{bmatrix}\lambda(z)/2i & \bar{q}\\ q & -\lambda(z)/2i\end{bmatrix} , \tag{A1}$$

$$V(x,t;z)=\begin{bmatrix}|q|^2-\lambda^2(z)/2i & -i\partial\bar{q}/\partial x-\lambda(Z)\bar{q}\\ -i\partial\bar{q}/\partial x-\lambda(z)\bar{q} & -|q|^2+\lambda^2(z)/2i\end{bmatrix} , \tag{A2}$$

where λ is a function of z, introduced in formula (21). One can straightforwardly check that the compatibility condition for the system (13) and (14)

$$U_t-V_x+UV-VU=0 \tag{A3}$$

(the so-called zero-curvature condition) is equivalent to the stable NSE. The scattering problem (13) with the matrix U given by (A1) possesses symmetry (involution) with respect to the inversion $z\to\omega^2/z$:

$$U(x,t;z)=U(x,t;\omega^2/z) \tag{A4}$$

(the dependence on time is omitted), which manifests itself in the properties of the Jost functions and the monodromy matrix. So, one can get from (A4) together with the boundary conditions (19) that

$$T_\pm(x,\omega^2/z)=\frac{z}{\omega}T_\pm(x,z)\sigma_2 . \tag{A5}$$

This relation leads to the following one for the monodromy matrix T defined by (15):

$$T(\omega^2/z)=\sigma_2 T(z)\sigma_2 , \tag{A6}$$

i.e., $T(z)$ may be represented as

$$T(z)=\begin{bmatrix}T_{11}(z) & -T_{21}(\omega^2/z)\\ T_{21}(z) & T_{11}(\omega^2/z)\end{bmatrix} . \tag{A7}$$

From this representation it is clearly seen that

the zeros of $T_{11}(z)$ and $T_{22}(z), z_k$ and $\tilde{z}_k$ correspondingly, are related by

$$\tilde{z}_k=\frac{\omega^2}{z_k} . \tag{A8}$$

Note also that the property (A5) implies

$$|F_{\pm}^{(i)}(x;\omega^2/z)\rangle = -(z^2/\omega^2)|F_{\pm}^{(j)}(x;z)\rangle \ , \tag{A9}$$

where $i \neq j$ and $i,j = 1,2$.

APPENDIX B

In the paper we use the following designations and relations

$$|u\rangle = \text{col}(u_1,u_2), \quad \langle u| = (-u_2,u_1) \ , \tag{B1}$$

$$\langle u|w\rangle = \det \begin{bmatrix} u_1 & w_1 \\ u_2 & w_2 \end{bmatrix} \ , \tag{B2}$$

$$|u\rangle\langle w| = \det \begin{bmatrix} -u_1w_2 & u_1w_1 \\ -u_2w_2 & u_2w_1 \end{bmatrix} \ , \tag{B3}$$

$$A^{\dagger} = (\det A)A^{-1} \ , \tag{B4}$$

$$(|a\rangle,|b\rangle)\begin{bmatrix} \langle c| \\ \langle d| \end{bmatrix} = |a\rangle\langle c| + |b\rangle\langle d| \ , \tag{B5}$$

introduced in [21], where useful properties of these vectors can be found.

APPENDIX C

The involution of the scattering data can also be written in terms of the modified Jost functions and monodromy matrix $\Psi_{\pm}$ and S:

$$\Psi_{\pm}(x,\omega^2/z) = \frac{z}{\omega}\Psi_{\pm}(x,z)\sigma_2 \ , \tag{C1}$$

$$S(\omega^2/z) = \sigma_2 S(z)\sigma_2 \ . \tag{C2}$$

Using the involution properties (C1) and (C2), one can derive the following relation between the matrices G and H defined by (43) and (44):

$$G^{\dagger}(\omega^2/z) = \frac{z}{\omega}H(z)\sigma_2 \ . \tag{C3}$$

Varying this identity and using the fact that $\det H = \Delta(z)S_{11}(z)$, one can obtain

$$[\delta HH^{-1}](z) + [G^{-1}\delta G](\omega^2/z) = \frac{\delta S_{11}(z)}{S_{11}(z)}I \ . \tag{C4}$$

This can be used to calculate the integrals around the points $\pm\omega$.

Noting that

$$I_{\pm} - \tilde{I}_{\pm} = \frac{1}{2\pi i}\frac{1}{\pm\omega - z}\int_{\gamma\pm} d\zeta\{\delta HH^{-1}(\zeta) + G^{-1}\delta G(\omega^2/\zeta)\} \tag{C5}$$

and applying (C4) one can obtain (57). Analogously, the involution properties (C1) and (C2) can be used to evaluate the integrals around $\gamma_{\pm}$ and $\tilde{\gamma}_{\pm}$ in (64) and (67). To this end note that the matrix functions $\Lambda_{0,1}$ and $\tilde{\Lambda}_{0,1}$ defined by (62) and (63) are related as follows:

$$\tilde{\Lambda}_0\left[\frac{\omega^2}{z}\right] = -\Lambda_0(z) \ , \tag{C6}$$

$$\tilde{\Lambda}_1\left[\frac{\omega^2}{z}\right] = -\Lambda_q^{\dagger}(z) \tag{C7}$$

[which, again results from (C1)]. Hence

$$\int_{\gamma\pm} d\zeta\Lambda_0(\zeta) - \int_{\tilde{\gamma}\pm} d\zeta\tilde{\Lambda}_0(\zeta) = \int_{\gamma\pm} d\zeta[\Lambda_0(\zeta) + \tilde{\Lambda}_0(\omega^2/\zeta)] = 0 \tag{C8}$$

and

$$\begin{aligned}\int_{\gamma\pm} d\zeta\Lambda_1(\zeta) - \int_{\tilde{\gamma}\pm} d\zeta\tilde{\Lambda}_1(\zeta) &= \int_{\gamma\pm} d\zeta[\Lambda_1(\zeta) + \tilde{\Lambda}_1(\omega^2/\zeta)] \\ &= \int_{\gamma\pm} d\zeta[\Lambda_1(\zeta) - \Lambda_1^{\dagger}(\zeta)] \ .\end{aligned} \tag{C9}$$

Noting that for any two columns ψ and ϕ

$$|\psi\rangle\langle\phi| - |\phi\rangle\langle\psi| = \langle\phi|\psi\rangle I \tag{C10}$$

and that

$$\langle T_{+}^{(2)}(\zeta)|T_{-}^{(1)}(\zeta)\rangle = -\Delta(\zeta)T_{11}(\zeta) \ , \tag{C11}$$

one can obtain from (C9), (62), and (63)

$$\int_{\gamma\pm} d\zeta\Lambda_1 - \int_{\tilde{\gamma}\pm} d\zeta\tilde{\Lambda}_1 = -\int_{\gamma\pm} d\zeta\frac{\delta T_{11}}{T_{11}}I = \frac{\pi}{i}\frac{\alpha_{\pm}}{T_{11}(\pm\omega)}I \tag{C12}$$

in the pure soliton case.

APPENDIX D

Using the definition of the matrix H [see (43) and (44)] one can write the matrix δHH^{-1} as follows:

$$\delta HH^{-1} = \left[|\delta T_{-}^{(1)}\rangle, |\delta T_{+}^{(2)}\rangle\right]\frac{1}{\Delta T_{11}}\begin{bmatrix} -\langle T_{+}^{(2)}| \\ \langle T_{-}^{(1)}| \end{bmatrix} \ . \tag{D1}$$

Employing the fact that

$$\frac{1}{T_{11}}|T_{-}^{(1)}\rangle = |T_{+}^{(1)}\rangle + r|T_{+}^{(2)}\rangle \ , \tag{D2}$$

where $r(z)$ is given by (28), one can represent $\delta HH^{-1} + \Lambda_1$ in the form

$$\delta HH^{-1} + \Lambda_1 = \frac{1}{\Delta}\{|\delta T_{+}^{(2)}\rangle\langle T_{+}^{(1)}| - |\delta T_{+}^{(1)}\rangle\langle T_{+}^{(2)}| - |\delta T_{+}^{(2)}\rangle\delta r\langle T_{+}^{(2)}|\} \ . \tag{D3}$$

Note that the first two terms inside the curly brackets are regular at the points z_k while for the last term z_k's are the poles of second order, since $\delta r = \delta T_{21}/T_{11} - T_{21}\delta T_{11}/T_{11}^2$. Using the formula

$$\mathrm{Res}_{z_k} \frac{f(z)}{T_{11}^2(z)} = \frac{1}{\dot{T}_{11}(z_k)} \left[\frac{d}{dz} \frac{f(z)}{\dot{T}_{11}(z)} \right]_{z=z_k} \tag{D4}$$

one can derive

$$\mathrm{Res}_{z_k}(\delta H H^{-1} + \Lambda_1) = \left[-|T_+^2\rangle \frac{\delta T_{21}}{\Delta \dot{T}_{11}} \langle T_+^2| + \frac{1}{\dot{T}_{11}} \frac{d}{dz} |T_+^2\rangle \gamma \delta T_{11} \langle T_+^2| \right]_{z=z_k} . \tag{D5}$$

Now some simple algebra leads to the result (70), where the quantities $\delta\gamma_k$ are defined by (30). The formula (71) can be obtained in a similar way.

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